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ANALYSIS OF LOCALIZED FAILURE IN ELASTOPLASTIC SOLIDS

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Abstract
The eigenanalysis of the localization tensor is essential for the analytical treatment of spatial discontinuities due to strain localization. The eigensolution characterizes the direction of the spatial discontinuity surface and the mode of localization, but not the magnitude of the jump in the strain rate. This paper describes the mathematical developments for detecting the onset of discontinuous bifurcation, and in addition it addresses the post-bifurcation analysis of elastoplastic solids. To this end constitutive expressions will be developed which relate the stress rate to the strain rate on both sides of the spatial discontinuity leading to a discussion of plastic/plastic vs. elastic/plastic bifurcation based on the eigenanalysis of a generalized localization tensor.
Keywords: Strain Localization, Elastoplastic Solid, Localization Tensor, Eigenanalysis, Discontinuous Bifurcation, Constitutive Relation.

1 Introduction

It has been widely accepted that quasi-brittle and ductile materials exhibit spatial discontinuities in the form of localization (NADAI (1950), HILL (1958)). The formation of cracks and shear bands in cementitious and granular materials, as well as in metals are typical examples of localized failure mechanisms.

Many researchers have been involved with this fascinating work and contributed to much progress in localization analysis. Their contributions may be divided into the construction of localization conditions "in the small", at the constitutive level (RUDNICKI AND RICE (1975), WILLAM AND SOBH (1987), BORRÉ AND MAIER (1989), BIGONI AND HUECKEL (1991), RUNESSON ET AL. (1991), OTTOSEN AND RUNESSON (1991)), and their finite element applications to reproduce strain localization "in the large", at the level of the boundary value problem (ORTIZ ET AL. (1987), DE BORST (1989), STEINMANN AND WILLAM (1991A), (1991B)). At the same time, it is important to establish a consistent failure theory which connects distributed failure on the continuum level with localized failure and discrete failure on the discontinuum level (WILLAM AND ETSE (1990), WILLAM ET AL. (1993)).

At the beginning of the present paper, the mathematical formulation of localization utilizing the localization tensor is reviewed. This well-established representation has usually been directed toward detecting the onset of strain localization as a condition of discontinuous bifurcation within the solid. To this end, it is fundamental to describe bifurcation phenomena via the second-order localization tensor which determines the orientation of the spatial discontinuity and the relative motion of the discontinuity, i.e. the mode of localized failure.

In the second point of this paper, we will extend the localization condition to capture also the post-bifurcation behavior. This leads to the eigenanalysis of the generalized localization tensor which permits quantitative discussion of plastic/plastic bifurcation vs. elastic/plastic bifurcation when different tangential constitutive descriptions govern the material behavior on both sides of the spatial discontinuity. The sensitivity of localization initiation with regard to variation of plastic properties will be investigated through perturbation analysis.

2 Review of Localization Criteria

2.1 Kinematics and constitutive law of discontinuity

Let us consider the homogeneously deformed solid subjected to quasi-static increments of deformation. In the bifurcated state, when the rate of displacement field $\dot{\mathbf{u}}$ is assumed to be continuous, the rate of displacement gradient field $\nabla\dot{\mathbf{u}}$ exhibits a jump across the discontinuity surface. This basic assumption describes a C^0-continuous motion where

$$[[\dot{\mathbf{u}}]] = \dot{\mathbf{u}}^+ - \dot{\mathbf{u}}^- = \mathbf{0}, \quad [[\nabla\dot{\mathbf{u}}]] = \nabla\dot{\mathbf{u}}^+ - \nabla\dot{\mathbf{u}}^- \neq \mathbf{0} \tag{1}$$

The superscripts " + "and " − " denote the values at the plus and minus side of the discontinuity surface, and the square brackets express the relative difference of a quantity across the discontinuity (Fig. 1).

The Maxwell compatibility condition requires that the jump of the displacement gradient has the form

$$[[\nabla\dot{\mathbf{u}}]] = \dot{\gamma}\mathbf{M}\otimes\mathbf{N} \tag{2}$$

where the unit normal vector $\mathbf{N}$ defines the orientation of the discontinuity surface and the unit vector $\mathbf{M}$ designates the direction of localized motion. It should be noted that the jump in the velocity gradient is a second order tensor of rank one. The indeterminate scalar $\dot{\gamma}$ denotes the amplitude of the jump in the velocity gradient, which is related to the wave speed in acoustic analysis.

The kinematic jump condition for the strain rate $[[\dot{\epsilon}]]$ is expressed as the symmetric part of the jump in the velocity gradient.

$$[[\dot{\epsilon}]] = \frac{1}{2}([[\nabla\dot{\mathbf{u}}]] + [[\nabla\dot{\mathbf{u}}]]^T) = \frac{\dot{\gamma}}{2}(\mathbf{M}\otimes\mathbf{N} + \mathbf{N}\otimes\mathbf{M}) = \dot{\gamma}[\mathbf{M}\otimes\mathbf{N}]^s \tag{3}$$

The kinematics of deformation across the discontinuity surface is thus represented by means of two vectors $\mathbf{M}, \mathbf{N}$ and the scalar $\dot{\gamma}$. Here, tensorial notation $\mathbf{x} \otimes \mathbf{y}$ indicates the tensor product of the vectors $\mathbf{x}$ and $\mathbf{y}$.

At the onset of the localization, the material response on both sides of the discontinuity surface is assumed to be in the plastic state. Hence, the constitutive characteristic for both sides are normally expressed by the same elastoplastic representation,

$$\dot{\boldsymbol{\sigma}}^{+} = \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}}^{+}, \quad \dot{\boldsymbol{\sigma}}^{-} = \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}}^{-} \tag{4}$$

$\mathbf{E}_{ep} = \mathbf{E}_e - \mathbf{E}_p$ denotes the elastoplastic tangent operator, comprised of the elastic stiffness $\mathbf{E}_e$ and the rank one modification $\mathbf{E}_p$ due to plasticity. $\mathbf{E}_{ep}$ is assumed to be identical on both sides of the discontinuity according to the concept of plastic/plastic bifurcation.

The discontinuity surface also causes a jump of the stress rate, denoted by $[[\dot{\boldsymbol{\sigma}}]] = \dot{\boldsymbol{\sigma}}^{+} - \dot{\boldsymbol{\sigma}}^{-}$. This stress jump is again related to the strain jump by means of the elastoplastic tangential operator,

$$[[\dot{\boldsymbol{\sigma}}]] = \mathbf{E}_{ep} : [[\dot{\boldsymbol{\epsilon}}]] \tag{5}$$

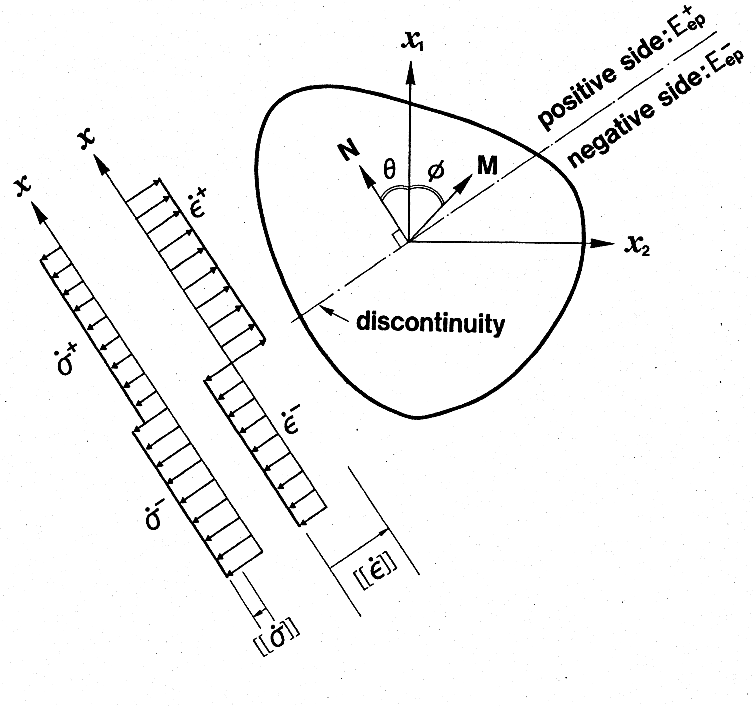

Fig. 1 Formation of Discontinuity Surface with Directional Vectors **M** and **N**.

2.2 Condition for onset of localization

Within the framework of discontinuity analysis in elastoplastic solids, the so-called localization operator $\mathbf{Q}_{ep}$ is defined by the second-order tensor:

$$\mathbf{Q}_{ep} = \mathbf{N} \cdot \mathbf{E}_{ep} \cdot \mathbf{N} \tag{6}$$

This second order tensor is identical with the characteristic tensor for discontinuous bifurcation or localization tensor. Although the stress jump appears across the discontinuity surface, the balance of linear momentum on the surface requires that the traction vector $\dot{\mathbf{t}}$ remains continuous i.e.,

$$[[\dot{\mathbf{t}}]] = \mathbf{N} \cdot [[\dot{\boldsymbol{\sigma}}]] = \mathbf{0} \tag{7}$$

Substitution of Eqs. (5) and (2) into the above equation leads to

$$\dot{\gamma}(\mathbf{N} \cdot \mathbf{E}_{ep} \cdot \mathbf{N}) \cdot \mathbf{M} = \dot{\gamma}\mathbf{Q}_{ep} \cdot \mathbf{M} = \mathbf{0} \tag{8}$$

This condition may be satisfied by the critical eigenvector $\mathbf{M}$ for a given discontinuity $\mathbf{N}$ in the case of localization. Localization initiates when a nontrivial solution of Eq. (8) exists with the condition that $\dot{\gamma} \neq 0$ and $\mathbf{M} \neq \mathbf{0}$. This necessitates that the localization tensor is singular, i.e.,

$$det(\mathbf{Q}_{ep}) = 0 \tag{9}$$

Otherwise, the magnitude of the strain jump $\dot{\gamma}$ must vanish: if $det(\mathbf{Q}_{ep}) \neq 0$, then $\dot{\gamma} = 0$. We conclude that the following conditions must hold for strain localization:

$$\begin{array}{rcllcl} det(\mathbf{Q}_{ep}) & = & 0 & \Longrightarrow \; \dot{\gamma} \neq 0, \; [[\dot{\epsilon}]] \neq 0 & \Longleftrightarrow & \text{localization} \\ det(\mathbf{Q}_{ep}) & \neq & 0 & \Longrightarrow \; \dot{\gamma} = 0, \; [[\dot{\epsilon}]] = 0 & \Longleftrightarrow & \text{no localization} \end{array} \tag{10}$$

Once the direction of discontinuity surface is determined, the corresponding vector $\mathbf{M}$ is calculated as the critical eigenvector of the localization tensor. The singularity condition of $\mathbf{Q}_{ep}$ is also called loss of ellipticity of the equilibrium equations, in analogy to the usual classification of the second order partial differential equation. The significance of conditions for discontinuous bifurcation may be understood in the following way. If a discontinuity surface was formed, this requires that the localization tensor must be singular for a particular direction $\mathbf{N}_{crit}$. The relative motion across the discontinuity surface must be prescribed by the corresponding eigenvector $\mathbf{M}$.

It is of importance to note that the statement so far is a pointwise argument for bifurcation of a material particle at the constitutive level. At the structural or element level, on the other hand, the localization condition must be dealt within a boundary value problem. This indicates that localization features such as shear band formation is one of the possible stationary states under the restricted kinematic compatibility conditions as well as the prescribed boundary conditions. We should emphasize that localization in the small does not necessarily lead to the failure of the structural element or the entire structure (Fig. 2).

This argument is purely deterministic concerning the formation of localized failure in a structure or an element. Clearly, initial imperfections and probabilistic scatter of material properties throughout the element or the structure play an important role to trigger formation of a spatial discontinuity. This is analogous to the classical instability arguments for elastic/inelastic buckling.

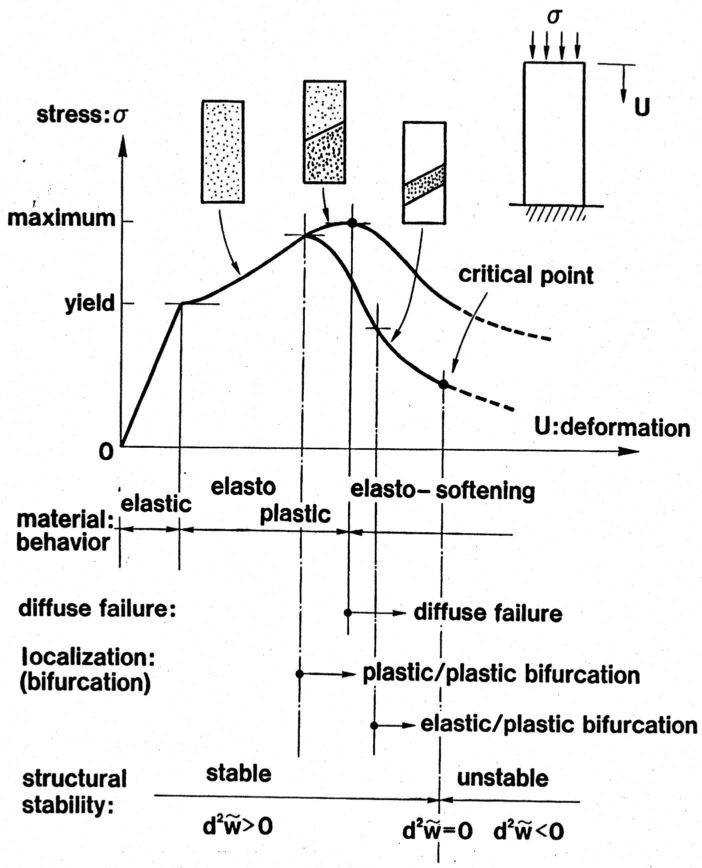

Fig. 2 Characteristics of Inelastic Behavior at Material and Element Level; Fundamental Failure Criteria during Deformation History

3 Extension to Post-Bifurcation Analysis

3.1 Different constitutive assumptions on each side of the discontinuity
Here, we try to expand the mathematical description of bifurcation beyond the onset of strain localization. To this end, this section will include identification of the magnitude of the strain rate jump, the kinematic relation of strain tensors on

both sides of the discontinuity, and the construction of constitutive representations between strain and stress tensors across the discontinuity surface.

The argument starts with the elastoplastic constitutive expressions which differ now on each side of the discontinuity surface (see Fig. 1) in contrast to Eq. (4).

$$\begin{aligned} \text{Positive Side}: \quad \dot{\boldsymbol{\sigma}}^{+} &= \mathbf{E}_{ep}^{+} : \dot{\boldsymbol{\epsilon}}^{+}, \quad \mathbf{E}_{ep}^{+} = \mathbf{E}_{ep}(\boldsymbol{\sigma}^{+}) \\ \text{Negative Side}: \quad \dot{\boldsymbol{\sigma}}^{-} &= \mathbf{E}_{ep}^{-} : \dot{\boldsymbol{\epsilon}}^{-}, \quad \mathbf{E}_{ep}^{-} = \mathbf{E}_{ep}(\boldsymbol{\sigma}^{-}) \end{aligned} \tag{11}$$

in which $\mathbf{E}_{ep}^{+}$ and $\mathbf{E}_{ep}^{-}$ denote the elastoplastic tangential operators on the positive and negative side of the discontinuity surface. This condition can readily be applied to the case of *elastic/plastic bifurcation* when we consider the special case $\mathbf{E}_{ep}^{+} \rightarrow \mathbf{E}_{ep}$ and $\mathbf{E}_{ep}^{-} \rightarrow \mathbf{E}_{e}$. Elastic/plastic bifurcation signifies that the material on one side of discontinuity surface unloads elastically, while the material on the other side loads plastically.

3.2 Constitutive expression analogous to plasticity

Now, let us consider the development of a constitutive representation of localization in the post-bifurcation regime analogous to classical plasticity theory.

The derivation for the bifurcated material response begins with three postulates following the well-known flow theory for rate-independent elastoplasticity.

1. Additive decomposition of strain rate: $\dot{\boldsymbol{\epsilon}}^{+} = \dot{\boldsymbol{\epsilon}}^{-} + [[\dot{\boldsymbol{\epsilon}}]]$

This decomposition should be considered in such a way that the strain rate tensor $\dot{\boldsymbol{\epsilon}}^{-}$ is the standard strain rate corresponding to the elastic strain rate and $\dot{\boldsymbol{\epsilon}}^{+}$ is the total strain rate with the jump of the strain rate $[[\dot{\boldsymbol{\epsilon}}]]$ being regarded as the plastic strain rate.

2. Flow rule describing evolution of localized strain jump: $[[\dot{\boldsymbol{\epsilon}}]] = \dot{\gamma}[\mathbf{M} \otimes \mathbf{N}]^{s}$

When $\dot{\gamma} \neq 0$ then localization is in effect and when $\dot{\gamma} = 0$ then no localization or elastic unloading takes place.

3. Yield condition for stress jump: $F = \mathbf{M} \cdot [[\dot{\mathbf{t}}]] = [\mathbf{M} \otimes \mathbf{N}]^{s} : [[\dot{\boldsymbol{\sigma}}]] = 0$

This corresponds to a scalar form of the traction balance across an arbitrary surface according to Cauchy.

It should be noted that the flow rule and the yield condition for discontinuous bifurcation render $d^2 W_{loc} = \frac{1}{2}[[\dot{\boldsymbol{\sigma}}]] : [[\dot{\boldsymbol{\epsilon}}]] = 0$. This states that the second order work of the stress and strain rate jumps must vanish for the formation of a spatial discontinuity.

Introducing the relation of stress rate jump and strain rate jump

$$\begin{aligned} [[\dot{\boldsymbol{\sigma}}]] &= \dot{\boldsymbol{\sigma}}^{+} - \dot{\boldsymbol{\sigma}}^{-} = \mathbf{E}_{ep}^{+} : [[\dot{\boldsymbol{\epsilon}}]] - [[\mathbf{E}_{p}]] : \dot{\boldsymbol{\epsilon}}^{-} \\ [[\dot{\boldsymbol{\sigma}}]] &= \dot{\boldsymbol{\sigma}}^{+} - \dot{\boldsymbol{\sigma}}^{-} = \mathbf{E}_{ep}^{-} : [[\dot{\boldsymbol{\epsilon}}]] - [[\mathbf{E}_{p}]] : \dot{\boldsymbol{\epsilon}}^{+} \end{aligned} \tag{12}$$

into the yield condition, the localization multipliers are readily determined in terms of the strain rate on either side of the discontinuity, yielding

$$\begin{aligned}
\dot{\gamma} &= \frac{[\mathbf{N}\otimes\mathbf{M}]^s : [[\mathbf{E}_p]] : \dot{\epsilon}^-}{\mathbf{M}\cdot\mathbf{Q}^+_{ep}\cdot\mathbf{M}} = \frac{[N_i M_j]^s [[E^p_{ijkl}]] \dot{\epsilon}^-_{kl}}{M_p Q^{ep+}_{pq} M_q} \\
\dot{\gamma} &= \frac{[\mathbf{N}\otimes\mathbf{M}]^s : [[\mathbf{E}_p]] : \dot{\epsilon}^+}{\mathbf{M}\cdot\mathbf{Q}^-_{ep}\cdot\mathbf{M}} = \frac{[N_i M_j]^s [[E^p_{ijkl}]] \dot{\epsilon}^+_{kl}}{M_p Q^{ep-}_{pq} M_q} \qquad (13)
\end{aligned}$$

in which

$$\begin{aligned}
[[\mathbf{E}_p]] &\equiv \mathbf{E}^+_p - \mathbf{E}^-_p = \mathbf{E}^-_{ep} - \mathbf{E}^+_{ep} \\
\mathbf{M}\cdot\mathbf{Q}^+_{ep}\cdot\mathbf{M} &= [\mathbf{M}\otimes\mathbf{N}]^s : \mathbf{E}^+_{ep} : [\mathbf{N}\otimes\mathbf{M}]^s \\
\mathbf{M}\cdot\mathbf{Q}^-_{ep}\cdot\mathbf{M} &= [\mathbf{M}\otimes\mathbf{N}]^s : \mathbf{E}^-_{ep} : [\mathbf{N}\otimes\mathbf{M}]^s \qquad (14)
\end{aligned}$$

This analytical consequence regarding $\dot{\gamma}$ provides the necessary information about the nature of the strain rate jump, which leads to a more fundamental understanding of the physical meaning of the discontinuity. If the tangential operators in both sides are the same, $\mathbf{E}^-_{ep} = \mathbf{E}^+_{ep}$, then the value of $\dot{\gamma}$ is equal to zero until the localization tensor is singular. Right at the moment of the onset of bifurcation, $\dot{\gamma}$ turns out to be indeterminate. This statement corresponds to the previous equation (8) signaling onset of strain localization. On the other hand, when discontinuities within the body develop so that the two tangential operators deviate, then $\mathbf{E}^-_{ep} \neq \mathbf{E}^+_{ep}$.

It also follows that the three strain rate tensors on both sides of the discontinuity $\dot{\epsilon}^+, \dot{\epsilon}^-$ and $[[\dot{\epsilon}]]$ are related as follows:

$$[[\dot{\epsilon}]] = \boldsymbol{\Phi} : \dot{\epsilon}^-, \qquad [[\dot{\epsilon}]] = \boldsymbol{\Psi} : \dot{\epsilon}^+ \qquad (15)$$

In this notation

$$\begin{aligned}
\boldsymbol{\Phi} &= \frac{[\mathbf{M}\otimes\mathbf{N}]^s \otimes [\mathbf{N}\otimes\mathbf{M}]^s : [[\mathbf{E}_p]]}{\mathbf{M}\cdot\mathbf{Q}^+_{ep}\cdot\mathbf{M}}, \\
\boldsymbol{\Psi} &= \frac{[\mathbf{M}\otimes\mathbf{N}]^s \otimes [\mathbf{N}\otimes\mathbf{M}]^s : [[\mathbf{E}_p]]}{\mathbf{M}\cdot\mathbf{Q}^-_{ep}\cdot\mathbf{M}} \qquad (16)
\end{aligned}$$

Both of $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are non-dimensional fourth-order tensors which provide the fundamental characteristics of discontinous bifurcation. Finally, the constitutive representations are

$$\dot{\boldsymbol{\sigma}}^+ = \left\{ \begin{array}{l} \mathbf{E}^+_{ep} : \dot{\epsilon}^+ \\ \mathbf{E}^+_{ep} : (\mathbf{I}_4 + \boldsymbol{\Phi}) : \dot{\epsilon}^- \end{array} \right. \qquad (17)$$

and similarly

$$\dot{\boldsymbol{\sigma}}^- = \left\{ \begin{array}{l} \mathbf{E}^-_{ep} : \dot{\epsilon}^- \\ \mathbf{E}^-_{ep} : (\mathbf{I}_4 - \boldsymbol{\Psi}) : \dot{\epsilon}^+ \end{array} \right. \qquad (18)$$

In view of each pair of these equations, it is obvious that two different strain rates are provided for the single stress rate as long as the tensors $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are defined. This is a fundamental aspect for the loss of uniqueness and post-bifurcation state due to the existence of a spatial discontinuity in the deformations. Fig. 3 schematically depicts the stress-strain behavior involving the bifurcation path and the relationship between the two strain tensors.

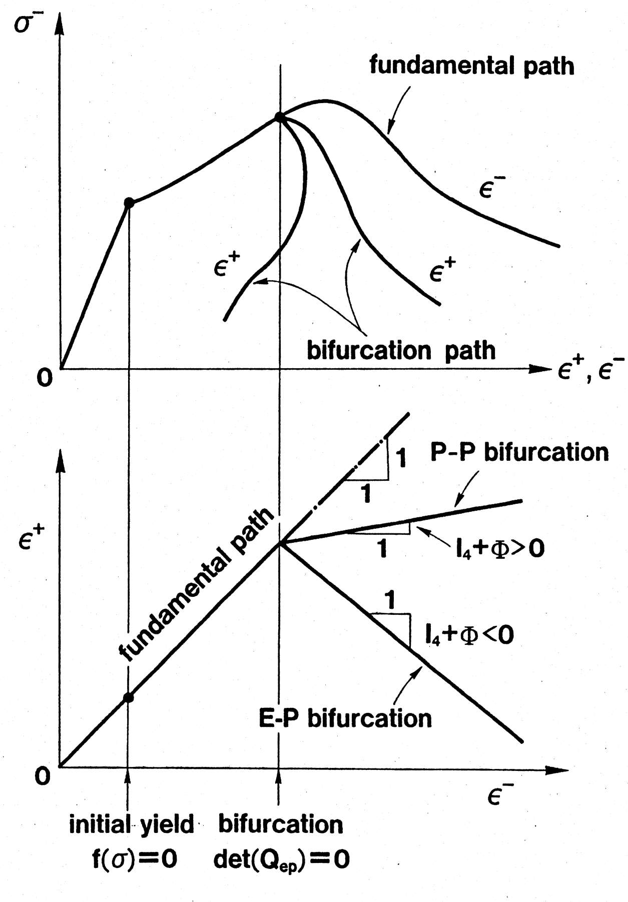

Fig. 3 Schematic Description of Fundamental/Bifurcation Paths (upper part); Relationship between Two Strain Tensors across Discontinuity (lower part)

When we assume the case of elastic/plastic bifurcation, for example, the stress rate on the elastic side may be related to the strain rate on the positive side as

$$\dot{\boldsymbol{\sigma}}^{-} = \{\mathbf{E}_e - \frac{\mathbf{E}_e : [\mathbf{M} \otimes \mathbf{N}]^s \otimes [\mathbf{N} \otimes \mathbf{M}]^s : \mathbf{E}_p^{+}}{[\mathbf{M} \otimes \mathbf{N}]^s : \mathbf{E}_e : [\mathbf{N} \otimes \mathbf{M}]^s}\} : \dot{\boldsymbol{\epsilon}}^{+} \quad (19)$$

This shows that the strain rate tensor on the plastic side (positive side in this case) is related to the stress tensor on the surrounding elastic side by a constitutive expression which is analogous to that of associated elastoplasticity,

$$\dot{\boldsymbol{\sigma}} = (\mathbf{E}_e - \mathbf{E}_p) : \dot{\boldsymbol{\epsilon}} = \{\mathbf{E}_e - \frac{\mathbf{E}_e : \mathbf{n} \otimes \mathbf{n} : \mathbf{E}_e}{H_p + \mathbf{n} : \mathbf{E}_e : \mathbf{n}}\} : \dot{\boldsymbol{\epsilon}} \tag{20}$$

where the scalar quantity H_p defines the plastic modulus. Making comparison with each other, one may realize that the second order tensor $[\mathbf{M} \otimes \mathbf{N}]^s$ of the discontinuity corresponds to the gradient of the yield function $\mathbf{n}$ in plasticity, and that the description for the bifurcated material has no counterpart with the plastic modulus.

4 Eigenanalysis of the Localization Tensor

In this section, we investigate the condition for discontinuous bifurcation based on conventional elastoplasticity through an analytical method. To this end, we first establish the generalized characteristic equation for the localization tensor in order to detect the bifurcation mode as well as the non-bifurcation mode. Then, analytic solutions of the eigenvalue problem are found and discussed, taking into consideration again different tangential operators on opposite sides. Furthermore, a perturbation analysis is carried out to examine the variation of the material properties.

4.1 Generalized characteristic equation and eigenanalysis

Let us expand the eigenvalue analysis allowing for different tangential operators on both sides of the discontinuity which are no longer identical. The discussion begins from Eq. (12), leading to

$$\dot{\gamma}\mathbf{Q}^{+}_{ep} \cdot \mathbf{M} = \mathbf{N} \cdot [[\mathbf{E}_p]] : \dot{\boldsymbol{\epsilon}}^{-}, \quad \dot{\gamma}\mathbf{Q}^{-}_{ep} \cdot \mathbf{M} = \mathbf{N} \cdot [[\mathbf{E}_p]] : \dot{\boldsymbol{\epsilon}}^{+} \tag{21}$$

At the instance of initial bifurcation, since $[[\mathbf{E}_p]] = \mathbf{0}$, the above equations turn out to be $\dot{\gamma}\mathbf{Q}_{ep} \cdot \mathbf{M} = \mathbf{0}$, which is the original expression of the characteristic equation. Here, the difference of tangential operators denoted by $[[\mathbf{E}_p]]$ can be developed as follows.

$$[[\mathbf{E}_p]] \equiv \mathbf{E}^{+}_p - \mathbf{E}^{-}_p = [[H_p]]\mathbf{E}_e : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}_e \tag{22}$$

in which the scalar quantity $[[H_p]]$ is defined $[[H_p]] \equiv \frac{1}{H^{+}_p + \mathbf{n}:\mathbf{E}_e:\mathbf{m}} - \frac{1}{H^{-}_p + \mathbf{n}:\mathbf{E}_e:\mathbf{m}}$ Then, we have

$$\frac{\dot{\gamma}\mathbf{Q}^{+}_{ep} \cdot \mathbf{M}}{\mathbf{n} : \mathbf{E}_e : \dot{\boldsymbol{\epsilon}}^{-}} = [[H_p]]\mathbf{N} \cdot \mathbf{E}_e : \mathbf{m}, \quad \frac{\dot{\gamma}\mathbf{Q}^{-}_{ep} \cdot \mathbf{M}}{\mathbf{n} : \mathbf{E}_e : \dot{\boldsymbol{\epsilon}}^{+}} = [[H_p]]\mathbf{N} \cdot \mathbf{E}_e : \mathbf{m} \tag{23}$$

in which $\mathbf{m}$ provides the direction of plastic deformation given as the gradient of potential surface and $\mathbf{n}$ the outward normal to the yield function. When $\mathbf{m} = \mathbf{n}$ is assumed, this reduces to the associated flow theory. The scalar value H_p is the plastic modulus expressing hardening and softening.

Equating two expressions in Eq. (23), one attains the equation in the form

$$\mathbf{Q}^{+}_{ep} \cdot \mathbf{M} \;=\; \lambda \mathbf{Q}^{-}_{ep} \cdot \mathbf{M}, \quad \text{where } \lambda = \frac{\mathbf{n} : \mathbf{E}_e : \dot{\epsilon}^{-}}{\mathbf{n} : \mathbf{E}_e : \dot{\epsilon}^{+}} \tag{24}$$

This can be regarded as the characteristic equation of a generalized eigenproblem in which λ denotes the eigenvalue and $\mathbf{M}$ the corresponding eigenvector. When one assumes $\mathbf{Q}^{-}_{ep} = \mathbf{Q}_e$, it reduces to the case which corresponds to solution by Runesson, Ottosen and Perić (1991).

Moreover, we would like to investigate the property of the eigenvalue λ. In order to eliminate the strain rate tensors in this expression , we again make use of

$$\dot{\epsilon}^{-} = (\mathbf{I}_4 - \boldsymbol{\Psi}) : \dot{\epsilon}^{+}, \quad \boldsymbol{\Psi} = \frac{[\mathbf{M} \otimes \mathbf{N}]^s \otimes [\mathbf{N} \otimes \mathbf{M}]^s : [[\mathbf{E}_p]]}{\mathbf{M} \cdot \mathbf{Q}^{-}_{ep} \cdot \mathbf{M}} \tag{25}$$

Then, Eq. (25) yields the critical eigenvalue λ^*.

$$\lambda^* \;=\; \frac{\mathbf{n} : \mathbf{E}_e : (\mathbf{I}_4 - \boldsymbol{\Psi}) : \dot{\epsilon}^{+}}{\mathbf{n} : \mathbf{E}_e : \dot{\epsilon}^{+}} = 1 - \frac{\mathbf{n} : \mathbf{E}_e : \boldsymbol{\Psi} : \dot{\epsilon}^{+}}{\mathbf{n} : \mathbf{E}_e : \dot{\epsilon}^{+}} \tag{26}$$

where λ^* is the eigenvalue associated with bifurcation. The case of $\boldsymbol{\Psi} = \mathbf{0}$, indicating the non-bifurcation mode, the above equation reduces to $\lambda = 1$.

Since the critical eigenvector $\mathbf{M}$ in the bifurcated mode is given as $\mathbf{M} \propto \mathbf{Q}_e^{-1} \cdot \mathbf{b}$, the critical eigenvalue can be finally represented as

$$\lambda^* \;=\; 1 - \frac{[[H_p]]\mathbf{a} \cdot (\mathbf{M} \otimes \mathbf{M}) \cdot \mathbf{b}}{\mathbf{M} \cdot \mathbf{Q}^{-}_{ep} \cdot \mathbf{M}} = \frac{1 - \frac{\mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b}}{H_p^{+} + c}}{1 - \frac{\mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b}}{H_p^{-} + c}} \tag{27}$$

Here, we introduced two vectors $\mathbf{a}$ and $\mathbf{b}$, and the scalar c as $\mathbf{a} = \mathbf{n} : \mathbf{E}_e \cdot \mathbf{N}, \quad \mathbf{b} = \mathbf{N} \cdot \mathbf{E}_e : \mathbf{m}, \quad c = \mathbf{n} : \mathbf{E}_e : \mathbf{m}$, which simplify, as an example, the notation of $\mathbf{Q}_p$ in such a way that $\mathbf{Q}_p \equiv \mathbf{N} \cdot \mathbf{E}_p \cdot \mathbf{N} = \frac{\mathbf{b} \otimes \mathbf{a}}{H_p + c}$. Full formulation process for these expressions may be referred in Yoshikawa (1993).

4.2 Critical hardening modulus for bifurcation

In addition to the derivation in the previous section, we reach the same result for the critical eigenvalue λ^* in Eq. (27) for the discontinuous bifurcation through alternative methods (Yoshikawa(1993)). Therefore, we describes the $\lambda^* = 0$ condition in terms of *the critical plastic modulus*, which indicates the condition for singularity of $\mathbf{Q}^{+}_{ep}$. This critical plastic modulus reads

$$H_p^{+} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} - c \quad \text{or} \quad H_p^{-} = -c \tag{28}$$

Likewise, one can also state that

$$H_p^{-} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} - c \quad \text{or} \quad H_p^{+} = -c \tag{29}$$

for $\mathbf{Q}_{ep}^{-}$, which can be derived from another characteristic equation $\mathbf{Q}_{ep}^{-} \cdot \mathbf{M} = \lambda' \mathbf{Q}_{ep}^{+} \cdot \mathbf{M}$.

The first equation in each of Eqs. (28) and (29) that $H_p^{crit} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} - c$ is identical to the result initially presented by RICE (1976) , and agrees with the eigenanalysis made by RUNESSON ET AL. (1991) based on the characteristic equation $\mathbf{Q}_{ep} \cdot \mathbf{M} = \lambda \mathbf{Q}_e \cdot \mathbf{M}$. Thus, this modulus H_p^{crit} is termed as *a standard critical plastic modulus* . The second condition in these equations $H_p = -c = -\mathbf{n} : \mathbf{E}_e : \mathbf{m}$ limits the elastoplastic tangential operator.

In order to distinguish the two bifurcation configurations, we can recast the eigenvalues for each case as follows.

$$\text{For Plastic/Plastic Bifurcation:} \quad H_p^{+} = H_p^{-} = H_p \implies$$
$$\lambda_{pp}^{*} = \begin{cases} 1 \text{ for } H_p \neq \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} - c, \\ \text{indeterminate for } H_p = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} - c \end{cases} \tag{30}$$
$$\text{For Elastic/Plastic Bifurcation:} \quad H_p^{+} = H_p, H_p^{-} = \infty \implies$$
$$\lambda_{ep}^{*} = 1 - (\frac{\mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b}}{H_p + c}) = 1 - \frac{\mathbf{n} : \mathbf{E}_e \cdot \mathbf{N} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{N} \cdot \mathbf{E}_e : \mathbf{m}}{H_p + \mathbf{n} : \mathbf{E}_e : \mathbf{m}} \tag{31}$$

where λ_{pp}^{*} denotes the eigenvalue pertinent to p-p bifurcation, while λ_{ep}^{*} that for e-p bifurcation.

Fig. 4 illustrates the variation of eigenvalue λ as a function of two plastic moduli H_p^{+} and H_p^{-}. This figure especially depicts the p-p bifurcation because the eigenvalue for $H_p^{-} \to \infty$ is far out side the figure. Fig. 5 indicates a comparison of p-p bifurcation and e-p bifurcation. The important finding is that the critical plastic modulus H_p^{crit} for $\lambda^{*} = 0$ is identical to each bifurcation configuration.

In these calculations, it is simply assumed that $\mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} = 3$ and $c = 2$ so that $H_p^{crit} = 1$ in order to simplify the numerical calculation. Hence, it means that this assumption is made regardless of the type of yield surfaces and potential surfaces, supposing that both of p-p and e-p bifurcation possess the same critical direction of discontinuity surface, i.e. $N = N^{crit}$.

Eigenvalue of Acoustic Tensor

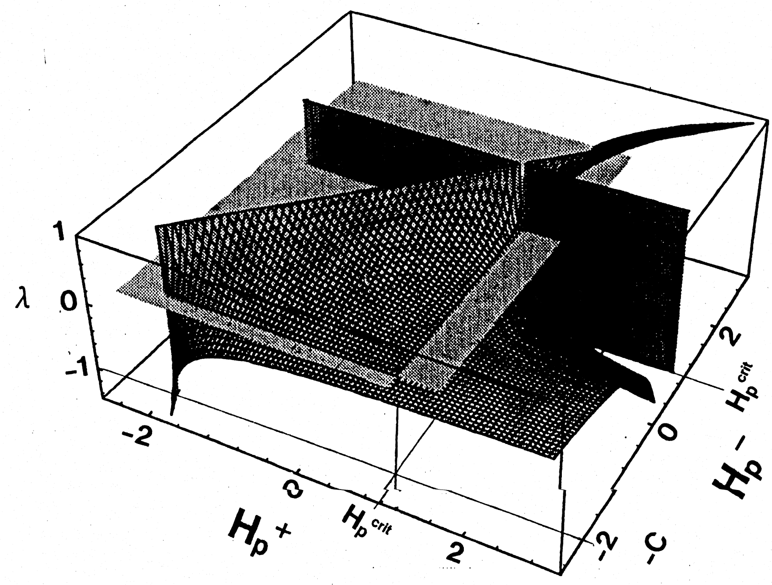

Fig. 4 Surface of Eigenvalue λ^* Expressed as a Function of Plastic Moduli H_p^+ and H_p^- on Each Side.

P-P Bifurcation v.s. E-P Bifurcation

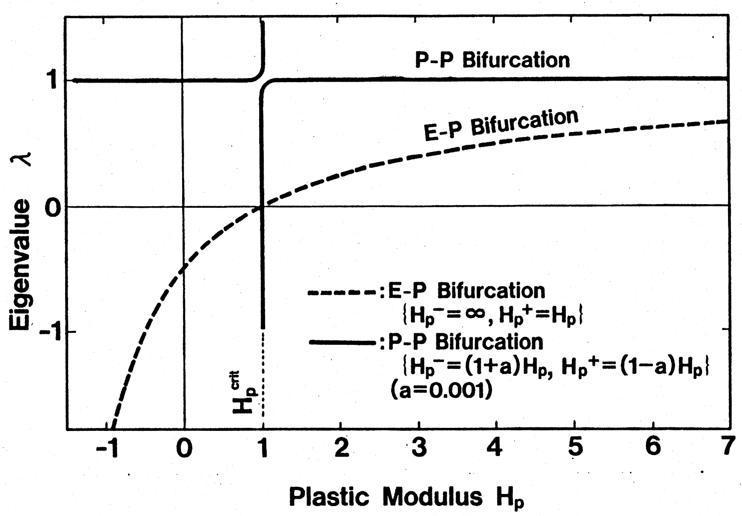

Fig. 5 Variation of Eigenvalue λ^* by Plastic Modulus H_p: Comparison of Plastic/Plastic and Elastic/Plastic Bifurcation

4.3 Perturbation analysis of plastic/plastic bifurcation

Next, we will investigate the effect of variations of plastic moduli across the discontinuity surface at the onset of bifurcation. This analysis can be carried out by imposing small differences of two plastic moduli on the bifurcation mode in order to see how the critical hardening modulus is influenced by this difference. This investigation may be regarded as *a perturbation analysis* with regard to the plastic property of the material across the discontinuity surface which initiates when $\lambda^* = 0$ is fulfilled. As an example of perturbation of the plastic properties on both sides of the discontinuity, we suppose

$$H_p^+ = (1-a)H_p, \quad H_p^- = (1+a)H_p \tag{32}$$

in which H_p denotes the mean value of the plastic modulus at the material particle under consideration, and a expresses the degree of perturbation of the plastic moduli on both sides which is assumed to vary between 0 to 1.

Shown in Fig. 6 are again the comparison of p-p bifurcation vs. e-p bifurcation, when it is assumed that $a = 0.005, 0.05, 0.5$. It is found that the critical moduli for p-p bifurcation are larger than that of e-p bifurcation which is identical to the standard critical value of plastic modulus. It depends on the difference of the two moduli, given by the perturbation factor of a. Consequently, we note that the bigger value of the plastic modulus implies that p-p bifurcation is initiated earlier than e-p bifurcation.

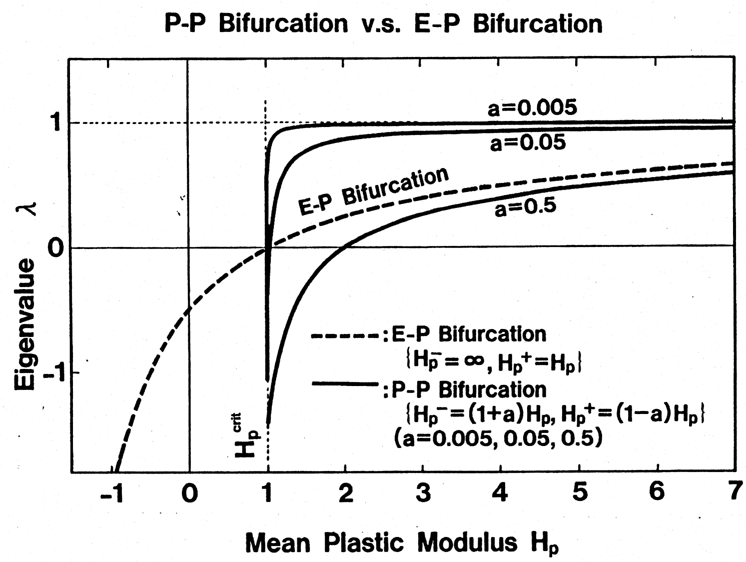

Fig. 6 Variation of Eigenvalue λ^* by Plastic Modulus H_p;
Perturbation Analysis of Plastic/Plastic Bifurcation with Assumption that
$H_p^+ = (1-a)H_p$, $H_p^- = (1+a)H_p$: Case for $a = 0.005, 0.05, 0.5$

5 Concluding Remarks

The present study describes mathematical formulations and their physical interpretation about fundamental aspects of failure mechanism and strain localization. Localization is regarded as a bifurcation problem which can be argued at a constitutive level.

The first point of this paper is that stress/strain relations and constraints of two strain tensors across the discontinuity surface can be represented in terms of non-dimensional fourth order tensors $\boldsymbol{\Phi}$ or $\boldsymbol{\Psi}$, which are characterized by vectors $\mathbf{M}, \mathbf{N}$ and the difference of elastoplastic tangential operators on both sides. These expressions describe the bifurcated path as well as the fundamental path at the constitutive level after the onset of bifurcation.

The second point is the eigenanalysis of the localization tensor based on the generalized characteristic equation in order to identify the bifurcation mode. The analytic solution of the critical eigenvalue and the corresponding critical value of plastic modulus lead to the discussion of plastic/plastic vs. elastic/plastic bifurcation, and perturbation analysis assigning slight differences among the plastic moduli across the discontinuity. One of the interesting findings is that the critical plastic modulus for plastic/plastic bifurcation is larger than that for elastic/plastic bifurcation, depending on how much the two plastic hardening/softening moduli differ from each other.

Some further discussion will be made as a next step to extends these concepts to the structural level considering a boundary value problem, assuming a volume fraction of localization region in a representative volume.

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